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Numerical solution for Option Pricing including two stocks by time fractional Black Shocles Equation.

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Abstract— The BS equations with fractional order two asset price model give the better prediction of options pricing in the monetary market, please see in [12]. In this paper, the changed form of BS-condition with two asset price models dependent on the Liovelle-Caputo derivative for good predictions of options prices is utilized. Finite volume Numerical Grid NODAL POINTS for solving finite volume Discretised Tri-Diagonal system of Linear Equations Iterative Technique is used to solve the equation.

Index Terms—Thomas Tri-Diagonal Matrix Algorithm (TDMA),options, Finite volume Method, time fractional ordered Black Sholes 2-Dimensional PDE for two stocks , Discredited system of Linear Equations.

1 Introduction

An option is an understanding between the proprietor and the purchaser to give the right to exchange a fixed number of shares of a predetermined regular stock at a fixed cost whenever at the latest a given date[1]. The demonstration of making this exchange is alluded to as practicing the option. The fixed cost is named the striking price at the given date[11]. A call choice gives the option to purchase the offers; a put alternative gives the option to sell the offers. The differential equation involving fractional order derivatives is a powerful tool for predicting the values of options. Options pricing is one of the foremost research areas in this context[12].

The possibility of BS model was first put out in " the Pricing of options and corporate Liabilities " of the journal of the political Economy by Fisher Black and Myron Sholes and then elaborated in “theory of Rational option pricing” by Robert Merton in 1973[12]. In 1979 Cox d. introduced the lattice method to calculate price options. In 1988 Structure, exhibit control variety produce to assess cost of alternative, which is like an explicit time-stepping scheme. In 1997, 24 years after the Black-Sholes model was first published by Myron Sholes and Robert Metron and they were awarded by Nobel Prize in Economics for a new method. In 2009, F. Geng, solved singular second order three-point boundary value problems using reproducing kernel Hilbert space method In 2010, Cen and Le introduced a robust finite difference scheme on a piece-wise uniform grids for American pricing put options[11]. In 2011, P. Bouboulis, M. Mavroforakis presented reproducing kernel Hilbert spaces and fractal interpolation for solving these non linear equations. In 2012, introducing uniform cubic B-Spline collocation Method. In 2014 and in 2015, Song Wang gave a finite volume method for discretization of Black Sholes Model to estimate price-options. In 2016, H. Zhang, F. Liu I. Turner, Q. Yang-solved time fractional Black–Scholes model governing equation for European options numerically. In 2019, D. Prathumwan and K. Trachoo solved Black Sholes equation by the method of Laplace Homotopy Perturbation Method for two asset European put option. In 2020, Zakaria, K. & Hafeez, S. the technique of Samudu transform method is used to demonstrate the analytical solution of 2-Dimensional, Time Fractional-ordered BS-Model, consists of two different assets in Liouville- Caputo Fractional derivative Form for the European call options. For the first time, the method of finite volume has been used for solving two assets BS financial model in the research work, which has provided better and approx same solution as obtained in other research works.

Methodology

FVM uses Grid-Nodal points Technique to solve PDE subjected to Initial, Boundary value condition. FVM is very simple easy and friendly to estimate Numerical solutions of PDE which are very closer to the exact solution of PDE. FVM depends upon three basic steps.

(i) The B.V is divided into the Network of Grid and Nodal points, at each nodal points we introduce infinitesima control voume

(ii) Calculate the volume of each nodal point by apply Integration over the control volume which provided discretised equation at each Nodal point.

(iii) Solve the system of linear equations or set of Discretised Linear equations by using TDMA.

Consider PD.E

$$\frac{\partial^{\propto}\emptyset}{\partial t^{\propto}}+\frac{\partial^2\emptyset}{\partial x^2}+\frac{\partial^2\emptyset}{\partial y^2}-\emptyset=0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

To get Numerical Sol^{ns} of above P.D.E, we divide the Boundary conditions into the network of small rectangular region or square of sides Δx and Δy, for our convenience letting $\Delta x = \Delta y = \delta x$ shown in figure. Assume$c_1, c_2, c_3, \ldots\ldots c_n$ be the given boundary conditions. East (e), West (w), North (n), South (s) with each internal node points.

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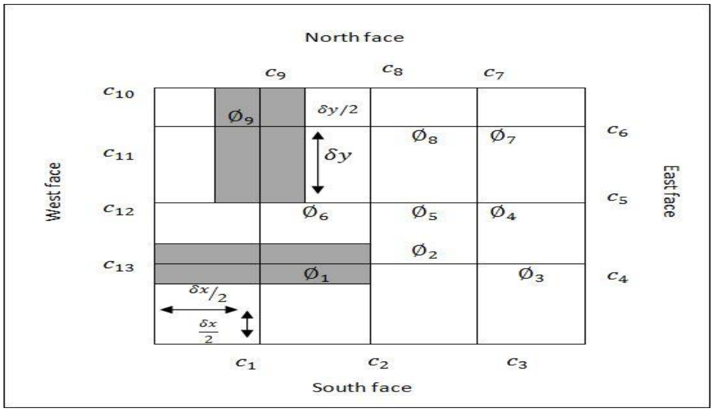

The above P.D.E (1) Can be integrated over control volume vas:

$$\int_V \frac{\partial^\propto \emptyset}{\partial t^\propto} dv + \int_V \frac{\partial}{\partial x}\left(\frac{\partial \emptyset}{\partial x}\right) dv + \int_V \frac{\partial}{\partial y}\left(\frac{\partial \emptyset}{\partial y}\right) dv$$

$$- \int_V \emptyset dv = 0$$

As $Ae = Aw = As = An = A[\therefore \Delta x = \Delta y = \delta x]$

$$\int_V \frac{\partial^\propto \emptyset}{\partial t^\propto} dxdydz + \int_V \frac{\partial}{\partial x}\left(\frac{\partial \emptyset}{\partial x}\right) dxdydz$$

$$+ \int_V \frac{\partial}{\partial y}\left(\frac{\partial \emptyset}{\partial y}\right) dxdydz$$

$$- \int_V \emptyset dxdydz = 0$$

$$\frac{\partial^\propto \emptyset}{\partial t^\propto} \delta x\, A + \left(\frac{\partial \emptyset}{\partial x} A\right)\Big|_e - \left(\frac{\partial \emptyset}{\partial x} A\right)\Big|_w + \left(\frac{\partial \emptyset}{\partial y} A\right) + \Big|_n$$

$$- \left(\frac{\partial \emptyset}{\partial y} A\right)\Big|_s - \emptyset \delta x A = 0$$

$$\frac{\emptyset p - \emptyset_{p^0}}{\Gamma(2-\propto)\Delta t^\propto} \delta x A + \frac{\partial \emptyset}{\partial x}\Big|_e A_e$$

$$- \frac{\partial \emptyset}{\partial x}\Big|_w A_w + \frac{\partial \emptyset}{\partial y}\Big|_n A_n + \frac{\partial \emptyset}{\partial z}\Big|_s A_s - \emptyset \delta x A$$

$$= 0$$

Flux Across through cell faces are:

Flux Across through west face $= A_w \frac{\partial \emptyset}{\partial x}\Big|_w =$

$A_w\left(\frac{\emptyset p - \emptyset_w}{\delta x_{wp}}\right) \ldots\ldots\ldots\ldots (i)$

Flux Across through south face $= A_s \frac{\partial \emptyset}{\partial x}\Big|_s =$

$A_s\left(\frac{\emptyset p - \emptyset_s}{\delta y_{sp}}\right) \ldots\ldots\ldots\ldots (ii)$

Flux Across through north face $= A_n \frac{\partial \emptyset}{\partial y}\Big|_n =$

$A_n\left(\frac{\emptyset n - \emptyset_p}{\delta x_{pe}}\right) \ldots\ldots\ldots\ldots (iii)$

Flux Across through east face $= A_e \frac{\partial \emptyset}{\partial x}\Big|_e =$

$A_e\left(\frac{\emptyset e - \emptyset_p}{\delta x_{pe}}\right) \ldots\ldots\ldots\ldots (iv)$

Substitute (i),(ii),(iii),(iv), Also $Ae = Aw = As = An = A[\therefore \Delta x = \Delta y = \delta x]$

$$(ii) \Rightarrow \left(\frac{\emptyset p - \emptyset p^0}{\Gamma(2-\propto)\Delta t^\propto}\right)\delta x + \left(\frac{\emptyset e - \emptyset_p}{\delta x_{pe}}\right) + \left(\frac{\emptyset p - \emptyset_w}{\delta x_{pw}}\right) + \left(\frac{\emptyset n - \emptyset_p}{\delta y_{np}}\right)$$

$$+ \left(\frac{\emptyset p - \emptyset_s}{\delta y_{sp}}\right) - \emptyset \delta x = 0$$

$$\left(\frac{\delta x}{\Gamma(2-\propto)\Delta t^\propto}\delta x - \frac{1}{\delta x_{pe}} - \frac{1}{\delta x_{pw}} - \frac{1}{\delta y_{np}} - \frac{1}{\delta y_{sp}}\right) - \emptyset_p + \left(\frac{1}{\delta x_{pe}}\right)\emptyset_e$$

$$+ \left(\frac{-1}{\delta x_{pw}}\right)\emptyset_w + \left(\frac{1}{\delta y_{np}}\right)\emptyset_n + \left(\frac{-1}{\delta y_{sp}}\right)\emptyset_s - \emptyset \delta x = 0 =$$

Substituting.

$$a_{p=}\frac{\delta x}{\Gamma(2-\propto)\Delta t^\propto}\delta x - \frac{1}{\delta x_{pe}} - \frac{1}{\delta x_{pw}} - \frac{1}{\delta y_{np}} - \frac{1}{\delta y_{sp}}$$

$a_{s\,=} {}^{-1}/_{\delta x_{sp}}$, $\quad a_{e\,=} {}^{1}/_{\delta x_{pe}}$, $\quad a_{w\,=} {}^{-1}/_{\delta x_{pw}}$

$$\boxed{a_p\emptyset_p + a_e\emptyset_e + a_w\emptyset_s + a_n\emptyset_n + a_s\emptyset_s - \emptyset\delta x = 0}$$

…………..(3)

(3) is the discretized system of linear equations. The discretised system of Linear equations for P.D.E (1) can be written as:

$$-a_s\emptyset_p + a_p\emptyset_p - a_n\emptyset_s = a_e\emptyset_e + a_w\emptyset_w + S$$

To solve the system of linear equations by TDMA , the discretised

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system of Linear Equations can be arranged as:

Let $C = a_e \emptyset_e a_w \emptyset_w + S$ where S = source term

Let us suppose $a_p = D_j$, $\propto_j = a_n$ and $\beta_j = a_s$

We have

$$\propto_j \emptyset_n + D_j \emptyset_p + \beta_j \emptyset_s = C_j \quad \text{--------------(4)}$$

The Tri-Diagonal system of Linear Equations can be written as:

$$\begin{aligned}
&\emptyset_1 = \emptyset_0\\
&-\beta_2 \emptyset_1 + D_2 \emptyset_2 - \propto_2 \emptyset_3 = 0\\
&-\beta_3 \emptyset_2 + D_3 \emptyset_3 - \propto_3 \emptyset_3 = 0 \qquad \text{(A)}\\
&-\beta_{n+1} \emptyset_n + D_n \emptyset_n - \propto_n \emptyset_n = 0
\end{aligned}$$

In above system of linear equation $\emptyset_1$ and $\emptyset_{n+1}$ demonstrates the boundary conditions.

From system of Linear Equations (A)

$$\emptyset_2 = \frac{\beta_2}{D_2}\emptyset_1 - \frac{\propto_2}{D_2}\emptyset_3 + \frac{C_2}{D_2} \quad \text{----------------- (i)}$$

$$\emptyset_3 = \frac{\beta_3}{D_3}\emptyset_2 - \frac{\propto_3}{D_3}\emptyset_4 + \frac{C_3}{D_3} \quad \text{---------------- (ii)}$$

TDMA consists of two Phases

(i) Forward Elimination Phase

(ii) Backward Substitution Phase

Forward Elimination Phase :

To eliminate $\emptyset_2$, pasting the value of $\emptyset_2$ in equation equation (ii) , we get the result.

$$\emptyset_3 = \frac{\propto_3 \emptyset_4}{D_3 - \frac{\propto_2}{D_2}\beta_3} + \frac{\beta_3\left[\frac{\beta_2}{D_2}\emptyset_1 + \frac{C_2}{D_2}\right] + C_3}{D_3 - \frac{\propto_2}{D_2}\beta_3}$$

Suppose $A_2 = \frac{\propto_2}{D_2}$ and $C_2' = \frac{\beta_2}{D_2}\emptyset_1 + \frac{C_2}{D_2}$

$\Longrightarrow$

$$\emptyset_3 = \frac{\propto_3 \emptyset_4}{D_3 - A_2\beta_3} + \frac{\beta_3 C_2' + C_3}{D_3 - A_2\beta_3}$$

Suppose $A_{3=} \frac{\propto_3}{D_3 - A_2\beta_3}$, $C_3' = \frac{\beta_3 C_2' + C_3}{D_3 - A_2\beta_3}$

$$\emptyset_3 = A_{3=} + C_3'$$

Generally we deduce the Algorithm

$$\emptyset_3 = A_n \emptyset_{n+1} + C_n' \quad \text{............(5)}$$

This is Forward Elimination Process.

(iii) Backward Substitution Phase:

$$A_o = 0$$

$$\beta_o = 0$$

$$A_j = \frac{\propto_j}{D_j - A_{j-1}\beta_j} \quad \text{............(6)}$$

$$C_j' = \frac{\beta_j C_{j-1}' + C_j}{D_j - A_{j-1}\beta_j} \quad \text{.............(7)}$$

We find all solutions by using equations (4) , (5) and (6)

By TDMA, Discretised system of Linear Equations (4) as

Type equation here.

$$-a_s\emptyset_s + a_p\emptyset_p - a_n\emptyset_n = a_e\emptyset_s + a_n\emptyset_n + S_t \quad \text{.........(4)}$$

S_t be source term

Eq.(4) Can be written Tri-Diagonal form as below

$$a_i\emptyset_i + b_i\emptyset_{i+1} + C_i\emptyset_{i-1} = d_i \quad \text{.............(7)}$$

$\Longrightarrow$ $-a_s\emptyset_p + a_p\emptyset_p - a_n\emptyset_n = d_i$ $[\ say\ d_i = a_e\emptyset_s + a_n\emptyset_n + S_i\ ]$

$$\begin{pmatrix}
a_1 & b_1 & 0 & 0 & 0 & 0\\
c_1 & a_2 & b_2 & 0 & 0 & 0\\
0 & c_2 & a_3 & b_3 & 0 & 0\\
0 & 0 & c_3 & a_4 & b_4 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & c_{n-1} & a_{n-1} & b_{n-1}\\
0 & 0 & 0 & 0 & c_n & a_n
\end{pmatrix}
\begin{pmatrix}
\emptyset_1\\ \emptyset_2\\ \emptyset_3\\ .\\ .\\ .\\ \emptyset_n
\end{pmatrix}
=
\begin{pmatrix}
d_1\\ d_2\\ .\\ .\\ .\\ d_n
\end{pmatrix}$$

Black Sholes Model , when price of two stocks are given.

$$\frac{\partial^{\propto} C}{\partial t^{\propto}} = -\frac{\sigma_1^2}{2}\frac{\partial^2 C}{\partial x^2} - \frac{\sigma_2^2}{2}\frac{\partial^2 C}{\partial y^2} - \rho\sigma_1\sigma_2\frac{\partial^2 C}{\partial x \partial y} + rC \text{ or}$$

$$\frac{\partial^{\propto} C}{\partial t^{\propto}} + \frac{\sigma_1^2}{2}\frac{\partial^2 C}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 C}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial^2 C}{\partial x \partial y} - rC = 0$$

Subject to Initial Boundary values conditions

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$C(x, y, 0) = Max(w_1 e^{x_1} + w_2 e^{y} - k\ , 0)$

- Exercise price of stock 1 = k_1
- Exercise price of stock 2 = k_2
- Maximum Exercise price for = k

  Strategy : Call option.
- Month of Expiration or time for Exercise data = t
- S.D of stock $1 = \sigma_1$
- S.D of stock $2 = \sigma_2$
- Proportion of stock 1 = w_1
- Proportion of stock 1 = w_2
- Risk free rate of return =r
- Correlation coefficient between Stock 1 and stock 2 = ρ

Given : $\sigma_{1} = 0.40$, $\sigma_{2} = 0.20$, r =8% , $\rho = 0.75$, $K = 60$, $w_1 = 1$, $w_2 = 3$; $\propto = 1$, 0.005, 0.25 , 0.5 and 0.75 and time =t =5 months

I.C : $C^o(x, y, 0) =$ Rs. 65

Boundary Conditions: (x and y are in Rs.)

C = Rs.70 at x = 0 ; $0 \le y \le 80$

C = Rs.60 at x = 60 ; $0 \le y \le 80$

C = Rs.50 at y = 0 ; $0 \le y \le 60$

C = Rs.20 at x = 80 ; $0 \le y \le 60$

Estimating call option prices by using Finite Volume Iterative Tri-Diagonal Algorithm Method (TDMA)

Setting the Network of grid and locate Node Or Mesh points.

For our convenience We scaling

$0 \le y \le 0.6,$

$0 \le y \le 0.8$,

$\delta x = \delta y = 0.2.$

$h_1 = \frac{\sigma_1^2}{2}$, $h_2 = \frac{\sigma_2^2}{2}$, $h_3 = -\rho\sigma_1\sigma_2$

C = 0.70 at x = 0 ; $0 \le y \le 0.8$

C = 0.60 at x = 0.6 ; $0 \le y \le 0.8$

C = 0.50 at y = 0 ; $0 \le y \le 0.6$

C = 0.20 at x = 0.8 ; $0 \le y \le 0.6$

Apply F.V.M

<u>At Node 1</u> :

$$h_1\left[\frac{C_e - C_p}{\delta x} - 0.70\right] + (h_2 + h_3)\left[\frac{C_n - C_p}{\delta y} - 0.5\right] - r\Delta x^2 C_p + \left(\frac{C_p - C_p^o}{\Delta t}\right)\Delta x^2 = 0$$

$$C_p\left[-h_1 - (h_2 + h_3) + r\Delta x^2 - \frac{\Delta x^2}{\Delta t}\right] + h_1 T_e + (h_2 + h_3)T_n = 0.70 h_1 \Delta x + (h_2 + h_3)(0.5)\Delta x + C_p^o \frac{\Delta x^2}{\Delta t}$$

$$0.3120C_p + 0.08a_e + 0.08a_n = 0.2117$$

<u>At Node 2</u> :

$$h_1\left[\frac{C_e - C_p}{\Delta x} - 0.70\right] + (h_2 + h_3)\left[\frac{C_n - C_p}{\Delta y} - \frac{C_n - C_s}{\Delta y}\right] - r\Delta x C_p + \left(\frac{C_p - C_p^o}{\Delta t}\right)\Delta x = 0$$

$$0.3120C_p + 0.08a_n + 0.08a_s + 0.08a_n = 0.2032$$

<u>At Node 3</u> :

$$0.08C_n + 0.3120C_p + 0.08a_s + 0.08a_e = 0.2032$$

<u>At Node 4</u> :

$$0.3120C_p + 0.08C_s + 0.08C_e = 0.1998$$

<u>At Node 5</u> :

$$0.08C_n + 0.3120C_p + 0.08C_w + 0.08a_e = 0.1920$$

<u>At Node 6</u> :

$$0.1472C_p + 0.08C_n + 0.08C_s + 0.08C_e + 0.08C_w = 0.1920$$

<u>At Node 7</u> :

$$0.1472C_p + 0.08C_n + 0.08C_s + 0.08C_e + 0.08C_w = 0.1920$$

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At Node 8 :

$$0.3120C_p + 0.08C_n + 0.08C_e + 0.08C_w = 0.1886$$

At Node 9 :

$$0.3120C_p + 0.08C_n + 0.08C_w = 0.1845$$

At Node 10 :

$$0.3120C_p + 0.08C_n + 0.08C_s + 0.08C_w = 0.1760$$

At Node 11 :

$$0.3120C_p + 0.08C_n + 0.08C_s + 0.08C_w = 0.1760$$

At Node 12 :

$$0.3120C_p + 0.08C_s + 0.08C_w = 0.1726$$

Computational Algorithm:

To solve BS Model by TDMA, system of Linear Equations can be settled along North-South line fashion as: $-a_s\emptyset_p + a_p\emptyset_p - a_n\emptyset_s = a_e\emptyset_e + a_w\emptyset_w + S$

Stopping Criteria of Iteration Process:

Repeat the Iteration process until a convergent solution is obtained

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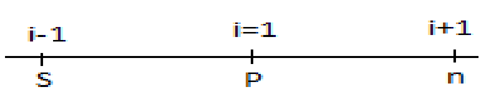

i.e $\quad C_{i+1} \cong C_i \qquad i = 1, 2, 3, \ldots$

By Tri-Diagonal Linear system of equations for solutions of BS PDE, the North-South Line- by-line fashion, the values of the coefficients are obtained

Nodes	a_p	a_n	a_s	a_e	a_w	s
1	0.3120	0.08	0	0.08	0	0.2117
2	0.3120	0.08	0.08	0.08	0	0.2032
3	0.3120	−0.08	0.08	0.08	0	0.2032
4	0.3120	0.08	0.08	0.08	0	0.1998
5	0.3120	0.08	0	0.08	0.08	0.2005
6	0.1472	0.08	0.08	0.08	0.08	0.1920
7	0.1472	0.08	0.08	0.08	0.08	0.1920
8	0.3120	0.08	0.08	0.08	0.08	0.1886
9	0.3120	0.08	0	0	0.08	0.1845
10	0.3120	0.08	0.08	0	0.08	0.1760
11	0.3120	0.08	0.08	0	0.08	0.1760
12	0.3120	0.08	0.08	0	0.08	0.1726

At 24th Iteration, solution becomes convergent up to four decimal places.

Nodes	C_j (Rs.)
1	91.58
2	83.24
3	10.932
4	30.6391
5	82.92
6	71.85
7	72.16
8	92.26
9	68.96
10	68.54
11	68.50
12	64.39

Hence values of call option pricing after 5 month is:

At $\propto = 1$

Price Of Stock 2 in (Rs.)	Price Of Stock 1 in (Rs.)		
	10	30	50
10	0.9158	0.8292	0.6896
30	0.8324	0.7185	0.6854
50	1.0932	0.7216	0.6850
70	30.6391	0.9226	0.6439

The table shows that the maximum gain may be obtained by selling the stock 1 at 10 and by selling the stock 2 at 70. The values for loss may also be depicted in the above table, If asset one is sold at 50 and asset two is sold at 70.

At $\propto = 0.005$

Price Of Stock 2 in (Rs.)	Price Of Stock 1 in (Rs.)		
	10	30	50
10	1.3571	1.1239	0.7600
30	1.1445	0.8490	0.7470
50	1.4562	0.8535	0.7444
70	15.1474	1.0047	0.6365

The table shows that the maximum gain may be obtained by selling the stock 1 at 10 and by selling the stock 2 at 70. The values for loss may also be depicted in the above table, If asset one is sold at 50 and asset two is sold at 70.

At $\propto = 0.25$

Price Of Stock 2 in (Rs.)	Price Of Stock 1 in (Rs.)		
	10	30	50
10	1.1477	0.9859	0.7263
30	0.9966	0.7851	0.7176
50	1.2859	0.7895	0.7163
70	19.9862	0.9659	0.64

The table shows that the maximum gain may be obtained by selling the stock 1 at 10 and by selling the stock 2 at 70. The values for loss may also be depicted in the above table, If asset one is sold at 50 and asset two is sold at 70.
For above diagram , it may easily be predicted, if stock 1 is sold at 10 and stock 2 is sold at 70, maximum profit may be obtained.

Conclusion & Discussion

By producing control volume grids at each nodal point, Finite Volume TDMA method provides successful and satisfactory numerical solutions of two dimensional Black Sholes time fractional ordered P.D.E for two stocks subjected to any initial-value boundary problems. The solution of Black Sholes P.D.E is efficiently and effectively estimated by simpler Finite Volume TDMA iterative scheme in least time with less computation. Illustrative practical example is also presented to understand the reliability, efficiency, simplicity, and effectiveness of the purposed scheme.

Acknowledgment

The authors declare that they have no conflict of interest.

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